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Algorithms for reconstruction with nondiffracting sources

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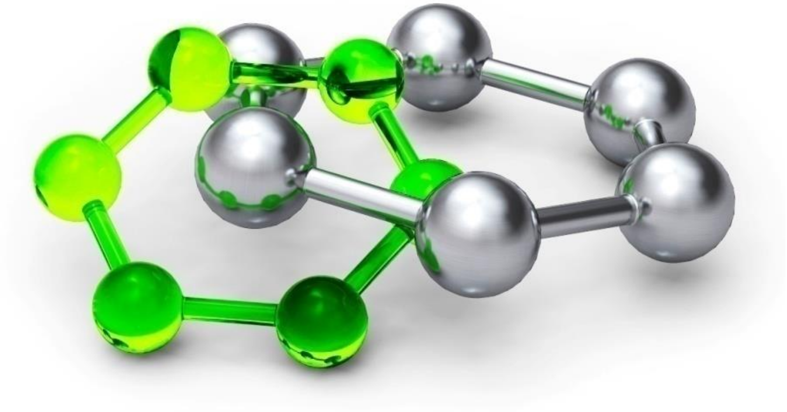
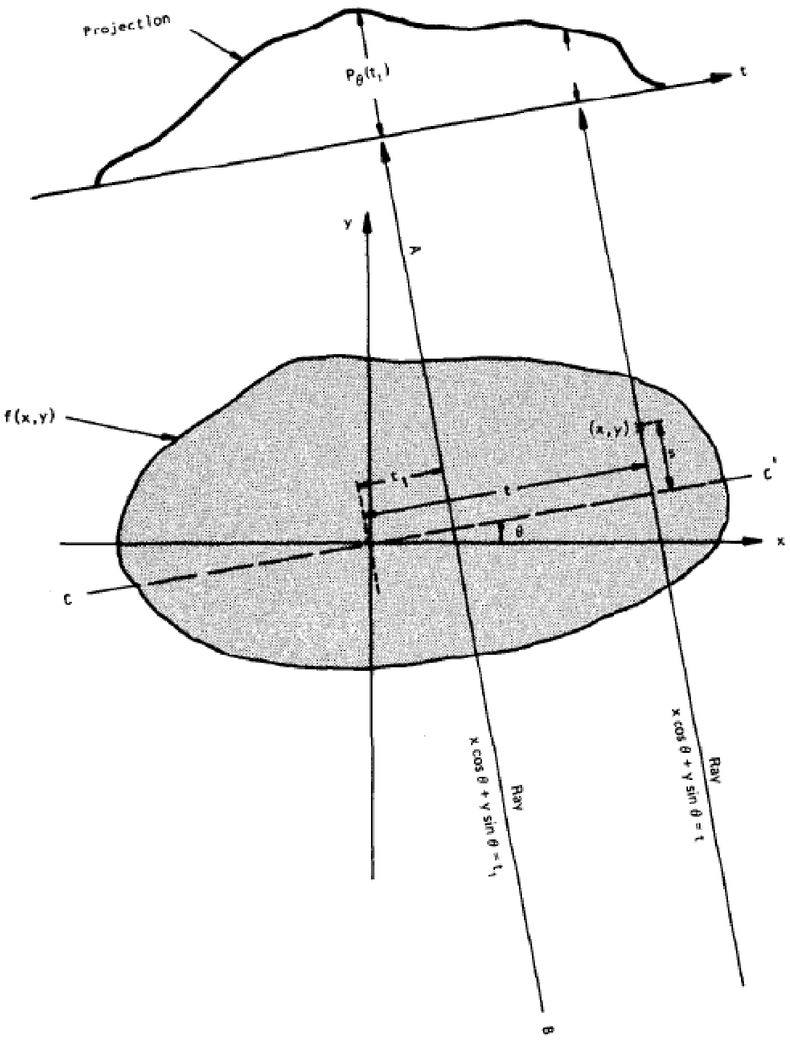


Line integrals and projections

- Line integral represents the total attenutaion of X-rays along a line through the object
- Function of the object *f(x,y)*
- Parametres of the line integral *(*θ*,t)*



- Equation of line AB:

$$x \cos\theta + y \sin\theta = t \qquad (1)$$

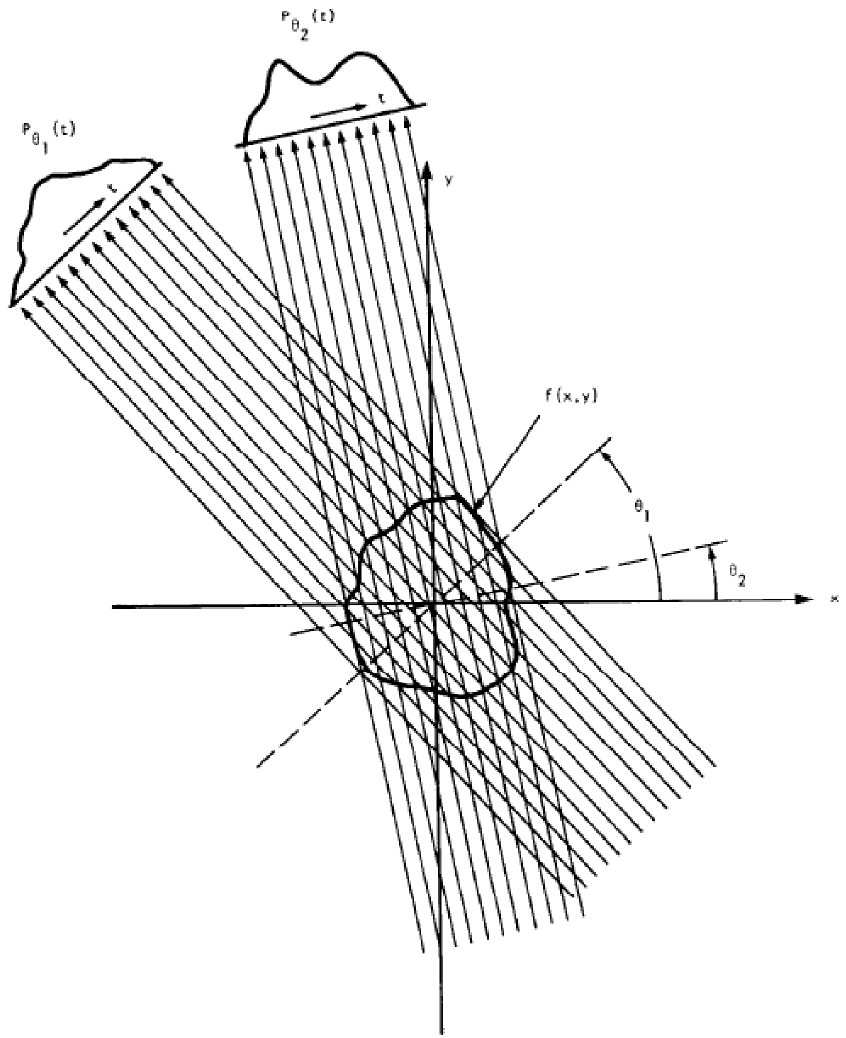
- Relationship of the line integral Pθ(t):

$$P_\theta(t) = \int_{(\theta,t)\ \text{line}} f(x, y)\, ds \qquad (2)$$

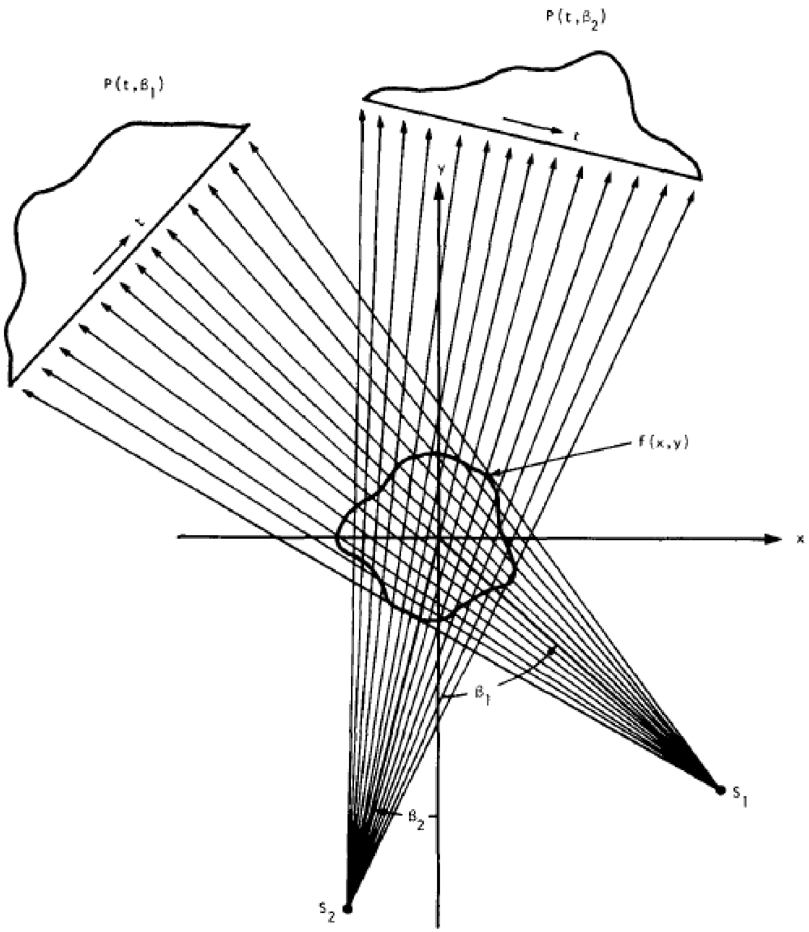
- Using delta function:

$$P_\theta(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,\delta(x \cos\theta + y \sin\theta - t)\, dx\, dy \qquad (3)$$

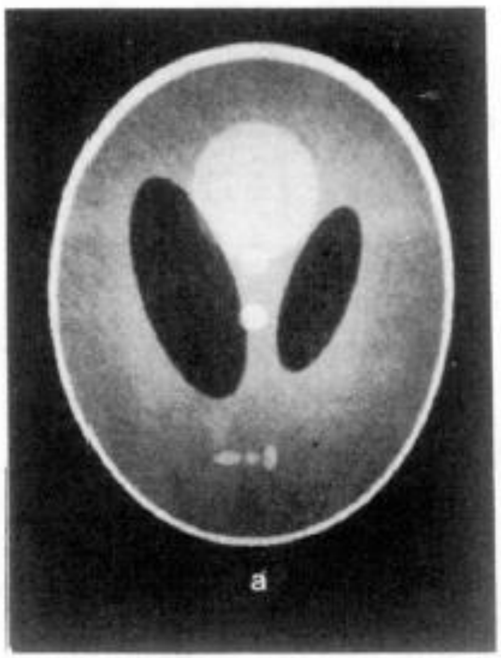
- Parallel projections taken by measuring of a set of parallel rays for a number of different angles



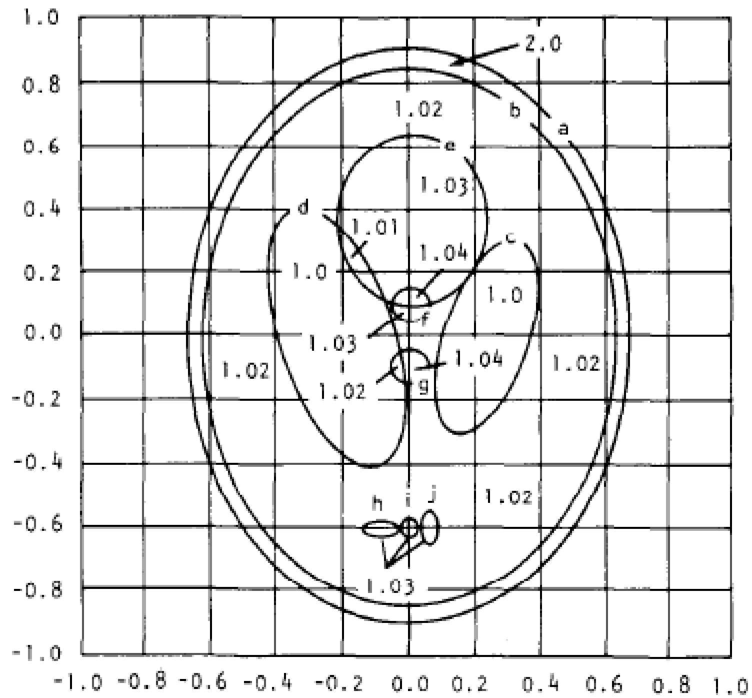
- A fan beam projection - single source is fixed relative to a line of detectors



- The Shepp and Logan „head phantom“
- Superposition of 10 ellipses



(a)



(b)

- Expressions for projections of a single ellipse:

$$f(x, y) = \begin{cases} \rho & \text{for } \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} \leq 1 \quad \text{(inside the ellipse)} \\ 0 & \text{otherwise} \quad \text{(outside the ellipse)} \end{cases} \tag{4}$$

- Expressions of projections:

$$P_\theta(t) = \begin{cases} \dfrac{2\rho AB}{a^2(\theta)} \sqrt{a^2(\theta) - t^2} & \text{for } |t| \leq a(\theta) \\ 0 & |t| > a(\theta) \end{cases} \tag{5}$$

The Fourier Slice Theorem

- Fourier transform of a parallel projection is equal to a slice of the two-dimensional Fourier transform of the original object -> derivation
- Two dimensional Fourier transform of the object:

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)}\, dx\, dy$$

(6)

- Projection at an angle θ, Pθ(t), and its Fourier transorm:

$$S_\theta(w) = \int_{-\infty}^{\infty} P_\theta(t) e^{-j2\pi wt}\, dt$$

(7)

- Projection at $\theta = 0$ and $v = 0$:

$$F(u, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi ux} \, dx \, dy \tag{8}$$

- Splitting integrals into two parts:

$$F(u, 0) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x, y) \, dy \right] e^{-j2\pi ux} \, dx \tag{9}$$

- Term in brackets as the equation for a projection:

$$P_{\theta=0}(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \tag{10}$$

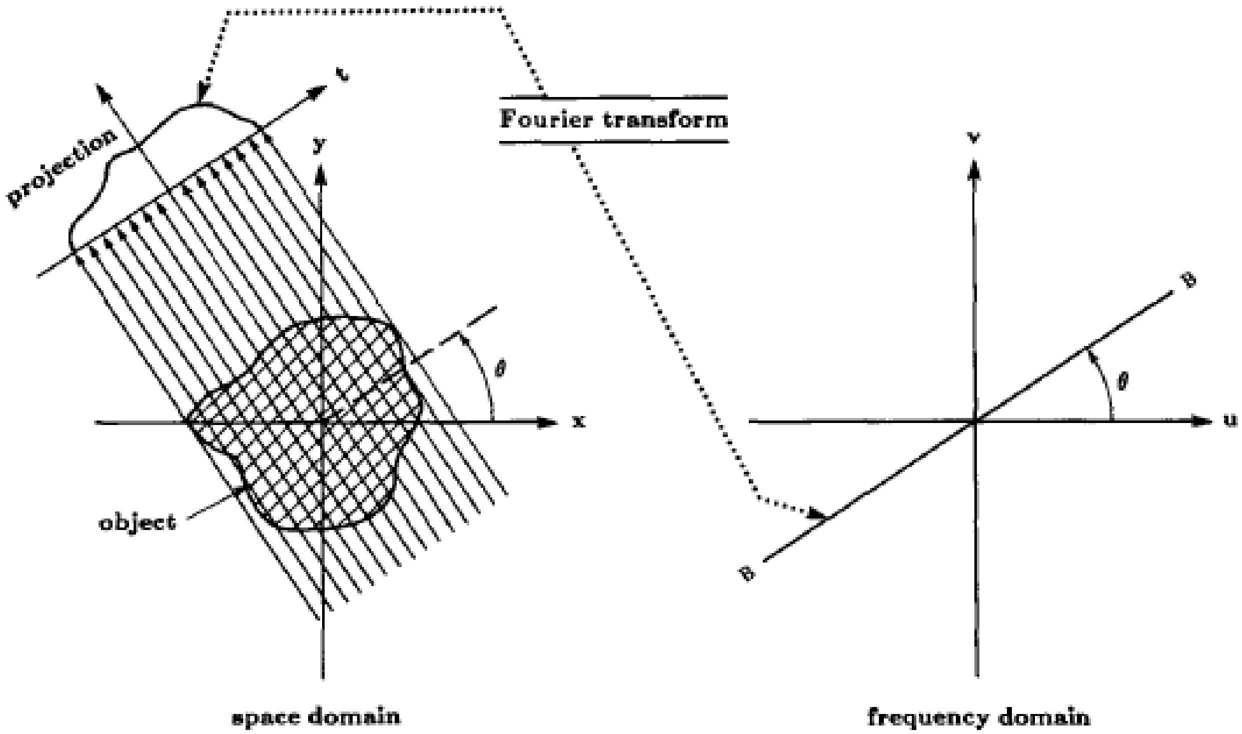
- Substituting this in (9):

$$F(u, 0) = \int_{-\infty}^{\infty} P_{\theta=0}(x) e^{-j2\pi ux}\, dx \tag{11}$$

- The simplest form of the Fourier Slice Theorem:

$$F(u, 0) = S_{\theta=0}(u) \tag{12}$$

- The Fourier Slice Theorem relates the Fourier transform of a projection to the Fourier transform of the object along a radial line



- *(t,s)* coordinate system as a rotated version:

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{13}$$

- Projection along lines of constant t:

$$P_\theta(t) = \int_{-\infty}^{\infty} f(t, s)\, ds \tag{14}$$

- Fourier transform:

$$S_\theta(w) = \int_{-\infty}^{\infty} P_\theta(t) e^{-j2\pi wt}\, dt \tag{15}$$

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- Substituting the projection:

$$S_\theta(w) = \int_{-\infty}^{\infty} \left[f(t, s)\, ds \right] e^{-j2\pi wt}\, dt \tag{16}$$

- Transforming into (x,y):

$$S_\theta(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi w(x \cos\theta + y \sin\theta)}\, dx\, dy \tag{17}$$

- Two-dimensional Fourier Transform at a spatial frequency ($u = w \cos\theta$, $v = w \sin\theta$):

$$S_\theta(w) = F(w, \theta) = F(w \cos\theta,\ w \sin\theta) \tag{18}$$

- Inverse Fourier transform:

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)}\, du\, dv \tag{19}$$

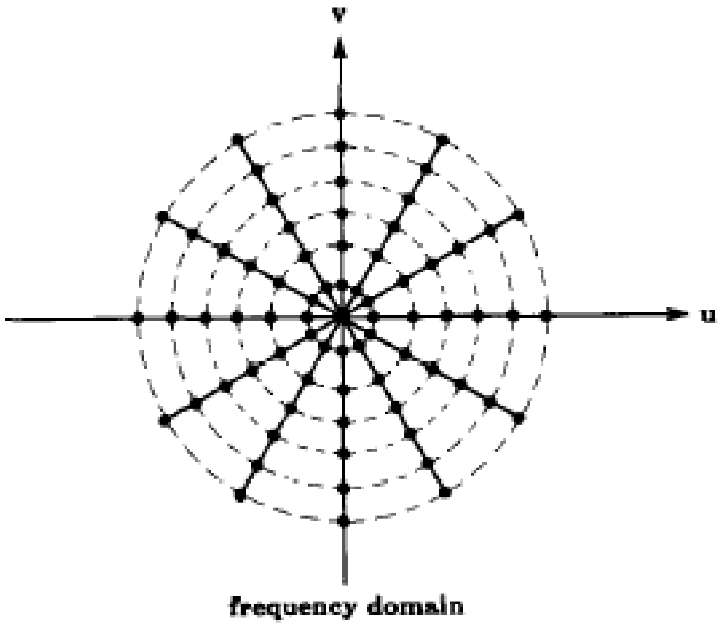
- For the purpose of computation:

$$f(x, y) \approx \frac{1}{A^2} \sum_{m=-N/2}^{N/2} \sum_{n=-N/2}^{N/2} F\left(\frac{m}{A}, \frac{n}{A}\right) e^{j2\pi((m/A)x+(n/A)y)} \tag{20}$$

- Previous equation for:

$$-\frac{A}{2} < x < \frac{A}{2} \text{ and } -\frac{A}{2} < y < \frac{A}{2} \tag{21}$$

- Collecting projections of the object at a number of angles gives estimates of the Fourier transform of the object along radial lines



Reconstruction algorithms for parallel projections

- The algorithm that is currently being used in almost all applications of straight ray tomography is the filtered backprojection algorithm.

- As the name implies, there are two steps to the filtered backprojection algorithm:

    - the filtering part, which can be visualized as a simple weighting of each projection in the frequency domain

    - the backprojection part, which is equivalent to finding the elemental reconstructions corresponding to each wedge filter mentioned above.

- The first step is:

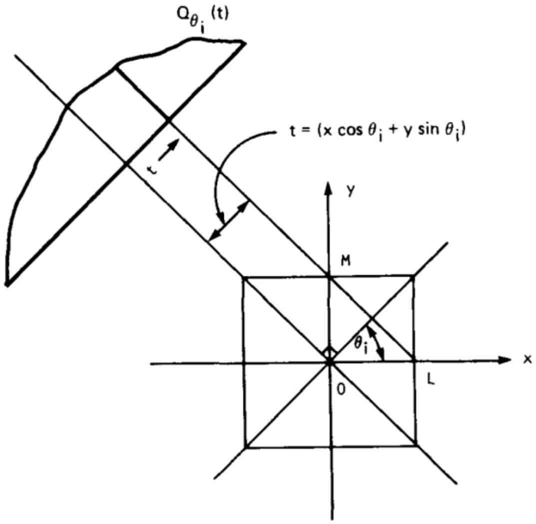
A simple weighting in the frequency domain is used to take each projection and estimate a pie-shaped wedge of the object's Fourier transform. Perhaps the simplest way to do this is to take the value of the Fourier transform of the projection, $S_\theta(w)$, and multiply it by the width of the wedge at that frequency.

Thus if there are K projections over 180° then at a given frequency w, each wedge has a width of $(2\pi|w|/K)\ S_\theta(w)$.

- The final reconstruction is found by adding together the two-dimensional inverse Fourier transform of each weighted projection. Because each projection only gives the values of the Fourier transform along a single line, this inversion can be performed very fast. This step is commonly called a backprojection.

- The complete filtered backprojection algorithm can therefore be written as:

1. Sum for each of the K angles, θ, between 0° and 180°
2. Measure the projection, $P_{\theta}(t)$
3. Fourier transform it to find $S_{\theta}(w)$
4. Multiply it by the weighting function $2\pi|w| / K$
5. Sum over the image plane the inverse Fourier transforms of the filtered projections (the backprojection process).



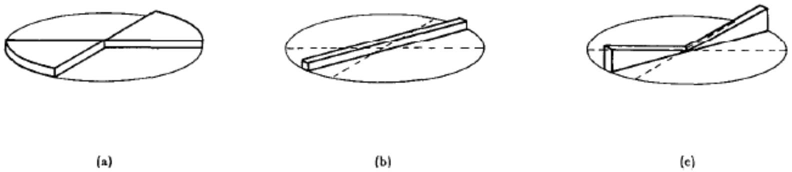


Fig. 3.8: *This figure shows the frequency domain data available from one projection. (a) is the ideal situation. A reconstruction could be formed by simply summing the reconstruction from each angle until the entire frequency domain is filled. What is actually measured is shown in (b). As predicted by the Fourier Slice Theorem, a projection gives information about the Fourier transform of the object along a single line. The filtered backprojection algorithm takes the data in (b) and applies a weighting in the frequency domain so that the data in (c) are an approximation to those in (a).*

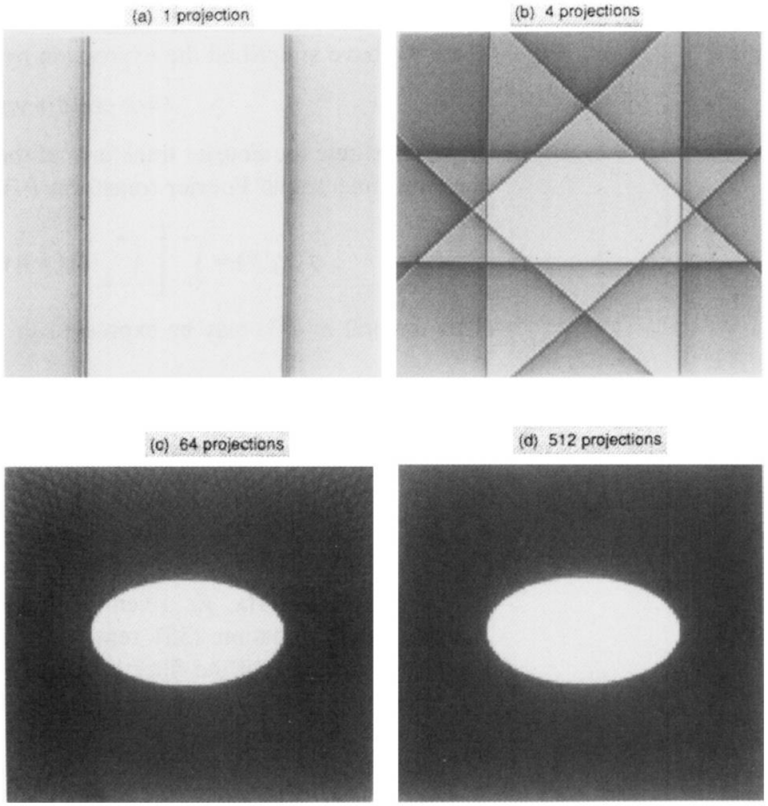


Fig. 3.10: *The result of backprojecting the projection in Fig. 3.9 is shown here. (a) shows the result of backprojecting for a single angle, (b) shows the effect of backprojecting over 4 angles, (c) shows 64 angles, and (d) shows 512 angles.*

Backprojection algorithm for parallel beam projections

- Recalling the formula for the inverse Fourier transform, the object function, f(x, y), can be expressed as:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}du\,dv$$

- we can write the inverse Fourier transform of a polar function as:

$$u = w\cos\theta \qquad v = w\sin\theta \qquad du\,dv = w\,dw\,d\theta$$

$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} F(w,\theta)e^{j2\pi w(x\cos\theta + y\sin\theta)}w\,dw\,d\theta$$

- Splitting the equation in two, considering θ from 0° to 180° and then from 180° to 360°

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} F(w,\theta)|w| e^{j2\pi wt}\, dw \right] d\theta$$

- If we substitute the Fourier transform of the projection at angle θ, $S_\theta(w)$ for the two-dimensional Fourier transform F(w, θ), we get:

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} S_\theta(w)|w| e^{j2\pi wt}\, dw \right] d\theta$$

- It can be expressed as:

$$f(x,y) = \int_0^{\pi} Q_\theta(x\cos\theta + y\sin\theta)\, d\theta \qquad Q_\theta(t) = \int_{-\infty}^{\infty} S_\theta(w)|w| e^{j2\pi wt}\, dw$$

- The parameter w has the dimension of spatial frequency. The integration in

$$Q_\theta(t) = \int_{-\infty}^{\infty} S_\theta(w)|w|e^{j2\pi wt}\,dw$$

- Must be carried out over all spatial frequencies. In practice the energy contained in the Fourier transform components above a certain frequency is negligible, so for all practical purposes the projections may be considered to be bandlimited.

If W is a frequency higher than the highest frequency component in each projection, then by the sampling theorem the projections can be sampled at intervals of

$$T = \frac{1}{2W}$$

- The FFT of $S_\theta(w)$:

$$S_\theta(w) \approx S\left(m\frac{2W}{N}\right) = \frac{1}{2W}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} P_\theta\left(\frac{k}{2w}\right)e^{-j2\pi(mk/N)}$$

- The next step is to evaluate the “modified projection” $Q_\theta(t)$ digitally. Since the Fourier transforms $S_\theta(w)$ have been assumed to be bandlimited (W), can be approximated by

$$Q_\theta(t) = \int_{-W}^{W} S_\theta(w)|w|e^{j2\pi wt}\,dw \quad \approx \quad \frac{2W}{N}\sum_{m=-\frac{N}{2}}^{\frac{N}{2}} S_\theta\left(m\frac{2W}{N}\right)\left|m\frac{2W}{N}\right| e^{j2\pi m(2W/N)t}$$

Filtering operation comments

$$Q_\theta(t) = \int_{-\infty}^{\infty} S_\theta(w)|w|e^{j2\pi wt}\,dw$$

- The last equation can be expressed in t-domain as:

$$Q_\theta(t) = \int P_\theta(\alpha)p(t-\alpha)d\alpha$$

- Where p(t) is nominally the IFT of the |w| function in the frequency domain. But in this case it is necessary to consider the all expression:

$$|w|e^{-\epsilon|w|} \xrightarrow{\text{The IFT}} p_\epsilon(t) = \frac{\epsilon^2 - (2\pi t)^2}{(\epsilon^2 + (2\pi t)^2)^2}$$

$$Q_\theta(t) = \int_{-\infty}^{\infty} j2\pi w S_\theta(w) \left[\frac{-j}{2\pi} sgn\ (w)\right] e^{j2\pi wt}\, dw \qquad sgn\ (w) = \begin{cases} 1 & for\ w > 0 \\ -1 & for\ w < 0 \end{cases}$$

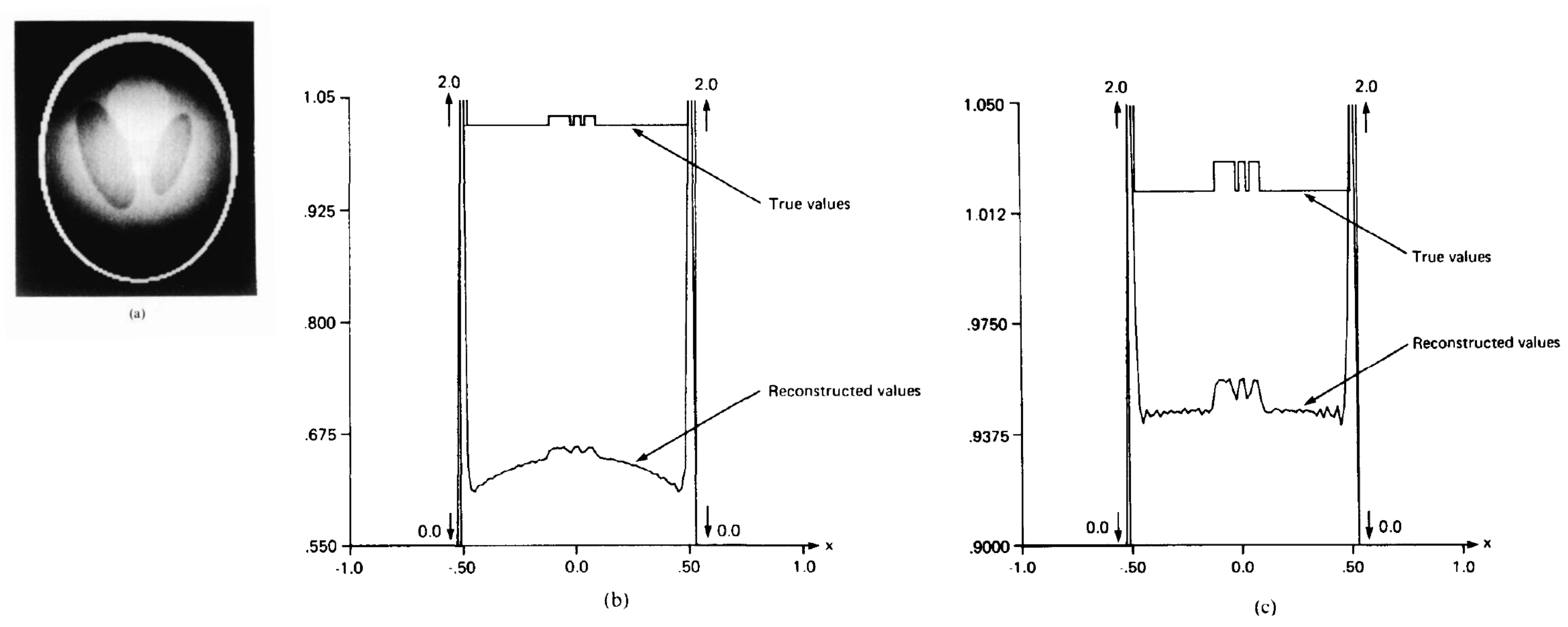
- By the standard convolution theorem:

$$Q_\theta(t) = \{IFT\ of\ j2\pi w S_\theta(w)\} * \left\{IFT\ of\ \frac{-j}{2\pi} sgn\ (w)\right\}$$

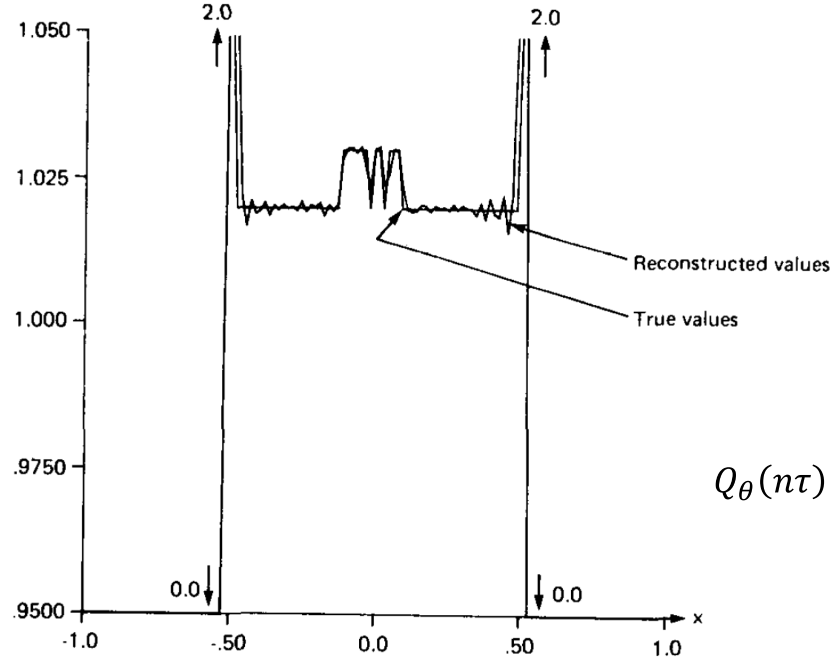
$$j2\pi w S_\theta(w) \xrightarrow{\text{IFT}} (\partial/\partial t) P_\theta(t) \qquad \frac{-j}{2\pi} sgn\ (w) \xrightarrow{\text{IFT}} \frac{1}{t}$$

$$Q_\theta(t) = \frac{1}{2\pi^2 t} * \frac{\partial P_\theta(t)}{\partial t} = Hilbert\ Transform\ of\ \frac{\partial P_\theta(t)}{\partial t}$$

Computer implementation of the Algorithm



(a)

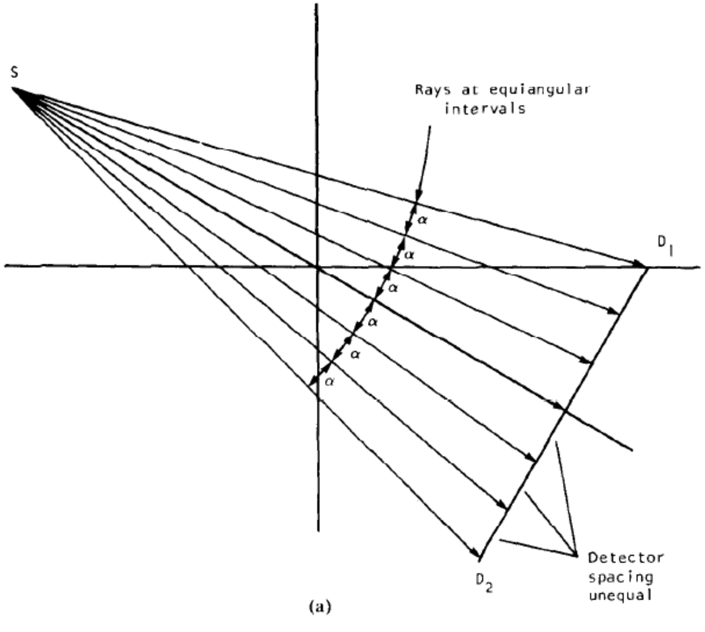


(b)

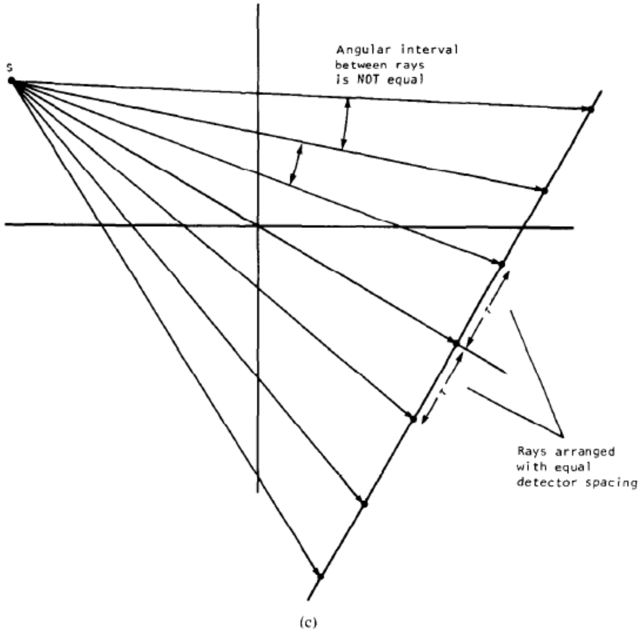
$$Q_\theta(n\tau) = \tau \times IFFT \begin{Bmatrix} [FFT\ P_\theta(n\tau)\ with\ ZP] \times \\ [FFT\ h(n\tau)\ with\ ZP] \times \\ smoothing\ - window \end{Bmatrix}$$

Reconstruction algorithms for fan projections

- Point source of radiation emanates a fan-shaped beam. The source and the entire bank of detectors, on the other side of scanned object, are rotated to generate the desired number of fan projections.
- Simple backprojection of parallel beam tomography ⟶ weighted backprojection of fan beam.
- Generally there are two types of fan projections:

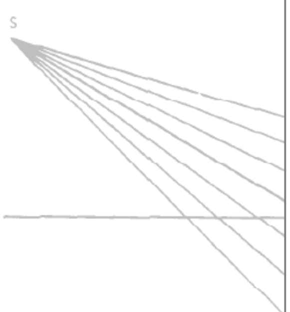


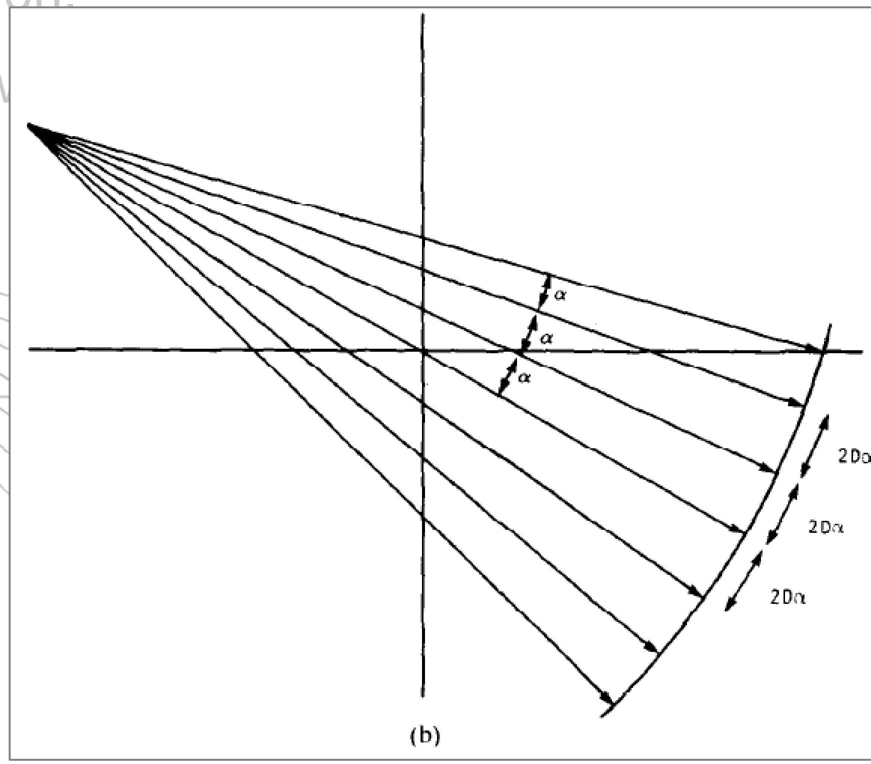
Equiangular

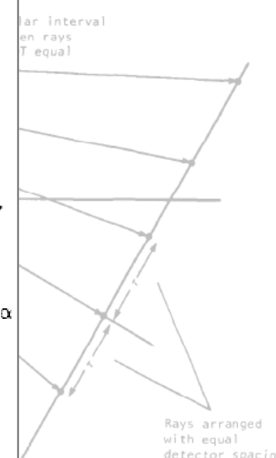


Equispaced

- Point source of radiation emanates a fan-shaped beam. The source and the entire bank of detectors, on the other side of scanned object, are rotated to generate the desired number of fan projections.
- Simple backprojection of parallel beam tomography → weighted backprojection.
- There are tw



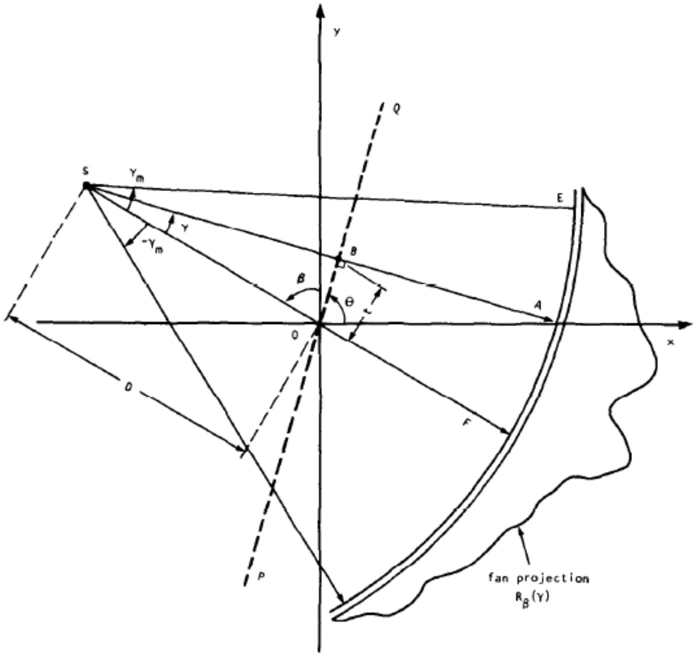


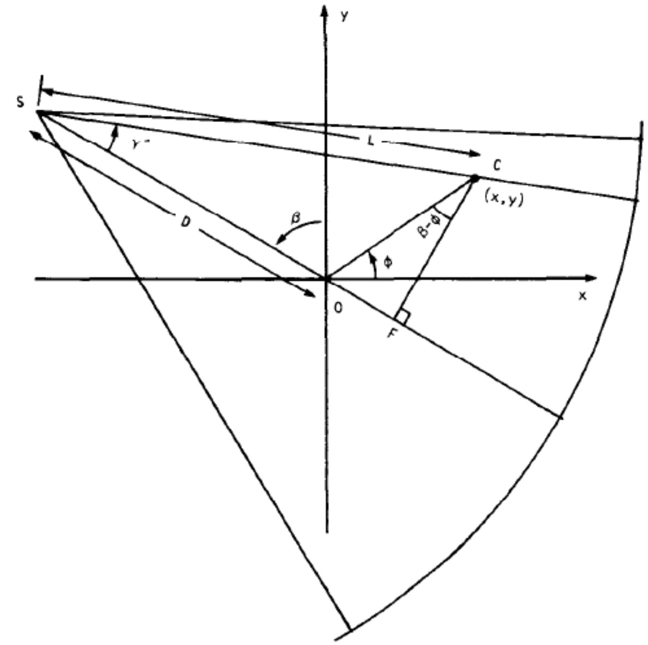


Equiangular

Equispaced

Equiangular rays

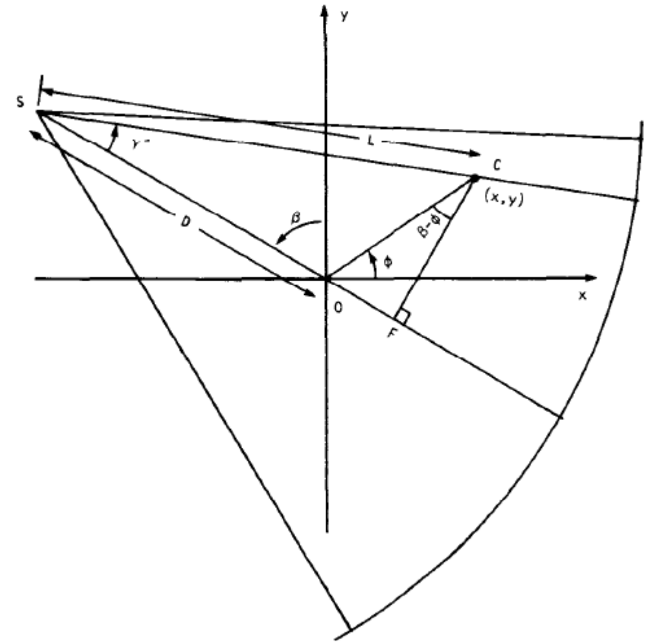




- $R_\beta(\gamma)$ is a fan projection and function $f(x, y)$, in polar coordinates $f(r, \phi)$ can be written as:

$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma) h(r\cos(\beta+\gamma-\phi) - D\sin\gamma) D\cos\gamma \, d\gamma d\beta$$

$$r\cos(\beta+\gamma-\phi) - D\sin\gamma) = L\,sin\,(\gamma' - \gamma)$$



- $R_b(\gamma)$ is a fan projection and function $f(r, \phi)$ in polar coordinates can be written as:

$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-\gamma m}^{\gamma m} R_\beta(\gamma) h(L \sin\ (L\sin(\gamma' - \gamma)\ \mathrm{D}\cos\gamma\, d\gamma d\beta$$

$$r\cos\ (\beta + \gamma - \phi) - D\sin\gamma) = L\sin\ (\gamma' - \gamma)$$

- The function *h(L sin (γ ʹ - γ))* will be then expressed in terms of *h(t)*. Note that *h(t)* is the inverse Fourier transform of |*w*| in the frequency domain:

$$h(t) = \int_{-\infty}^{\infty} |w|\, e^{j2\pi wt}\, dw \longrightarrow h(L \sin\ \gamma) = \int_{-\infty}^{\infty} |w|\, e^{j2\pi wL\, \sin\gamma}\, dw$$

$$\Downarrow$$

$$w' = w\frac{L \sin\gamma}{\gamma}$$

- Weighted filtered backprojection algorithm can be written as:

$$\boldsymbol{f(r,\phi) = \frac{1}{2}\int_{0}^{2\pi} \frac{1}{L^2}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma) g(\gamma' - \gamma) D \cos\gamma\, d\gamma d\beta}$$

- where

$$g(\gamma) = \frac{1}{2}\left(\frac{\gamma}{sin\,\gamma}\right)^2 h(\gamma)$$

- The image reconstruction is done using the following three steps:
  - Step 1 Each projection $R_\beta(\gamma)$ is sampled with sampling interval α. The known data then are $R_{\beta i}(n\alpha)$ where n takes integer values. The first step is to generate for each fan projection $R_{\beta i}(n\alpha)$ the corresponding $R'_{\beta i}(n\alpha)$:

$$R'_{\beta i}(n\alpha) = R_{\beta i}(n\alpha) \cdot D \cos n\alpha$$

  - Step 2 Discrete convolution using an FFT program is performed. Each modified projection $R'_{\beta i}(n\alpha)$ with $g(n\alpha)$ is convolved to generate the corresponding filtered projection:

$$Q_{\beta i}(n\alpha) = R'_{\beta i}(n\alpha) \cdot g(n\alpha)$$

Response for the discrete impulse

$$g(n\alpha) = \begin{cases} \dfrac{1}{8\alpha^2}, & n = 0 \\ 0, & n \text{ is even} \\ \left(\dfrac{\alpha}{\pi\alpha \sin n\alpha}\right)^2, & n \text{ is odd.} \end{cases}$$

  - Step 3 Weighted backprojection of each filtered projection along the fan is performed:

$$f(x,y) \approx \Delta\beta \sum_{i=1}^{M} \frac{1}{L^2(x,y,\beta_i)} Q_{\beta i}(\gamma')$$

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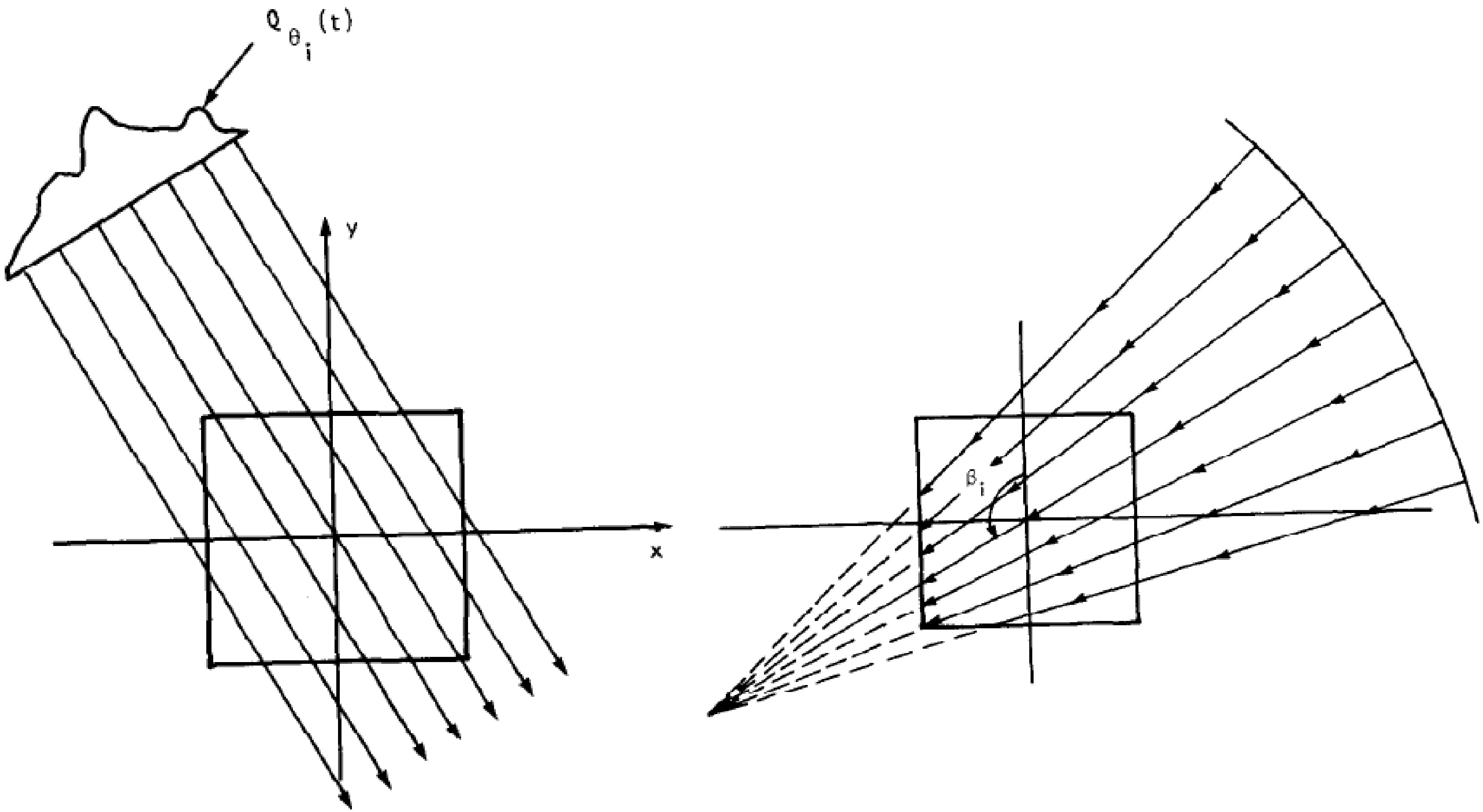
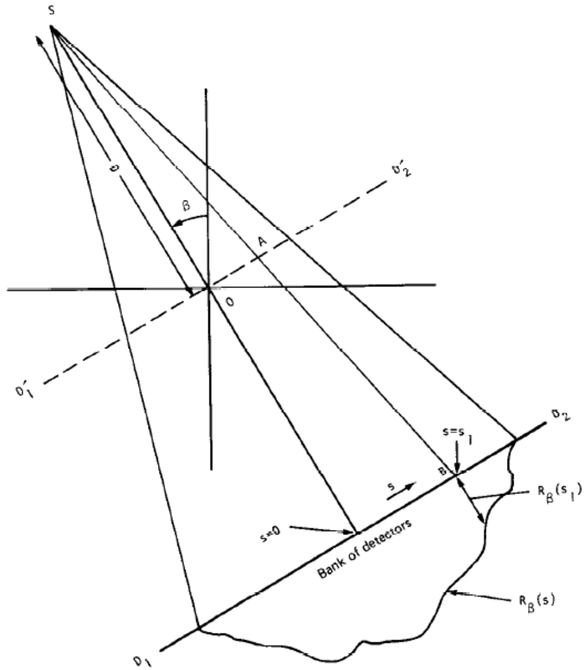
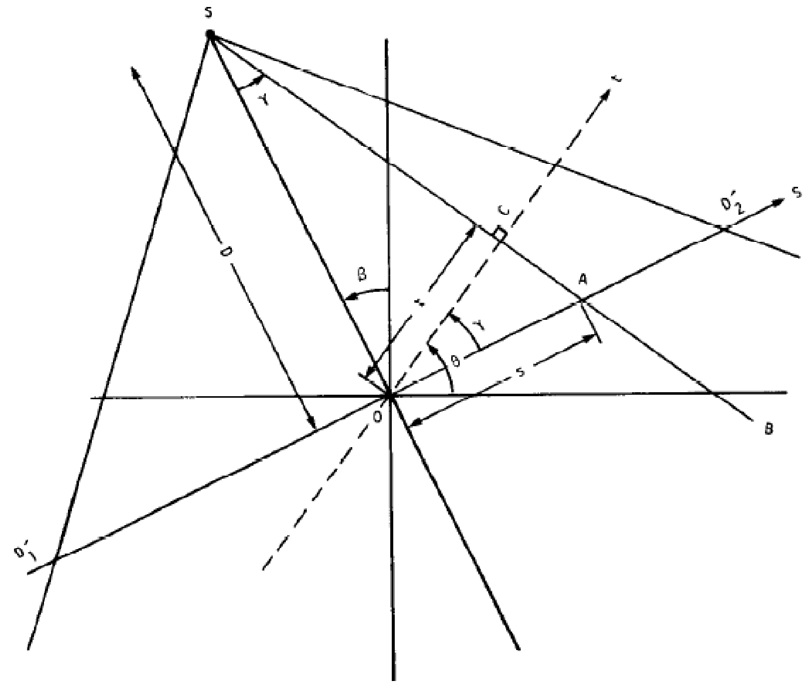


$$f(x,y) \approx \Delta\beta \sum_{i=1}^{M} \frac{1}{L^2(x,y,\beta_i)} Q_{\beta i}(\gamma')$$

Equally spaced collinear detectors

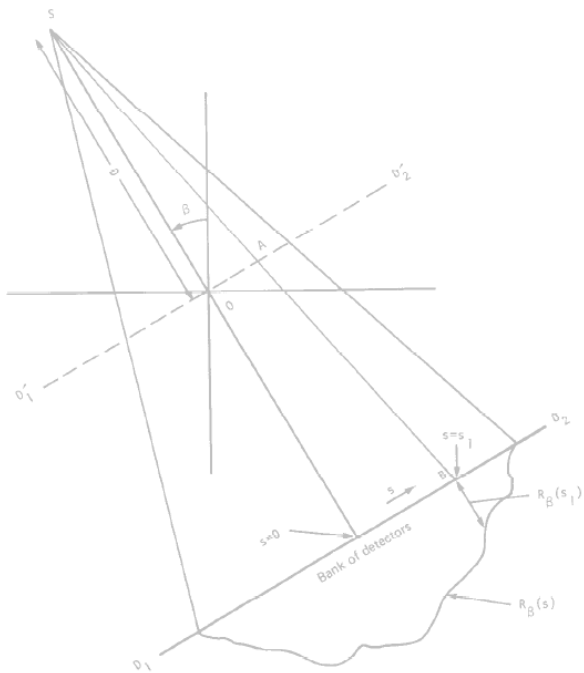


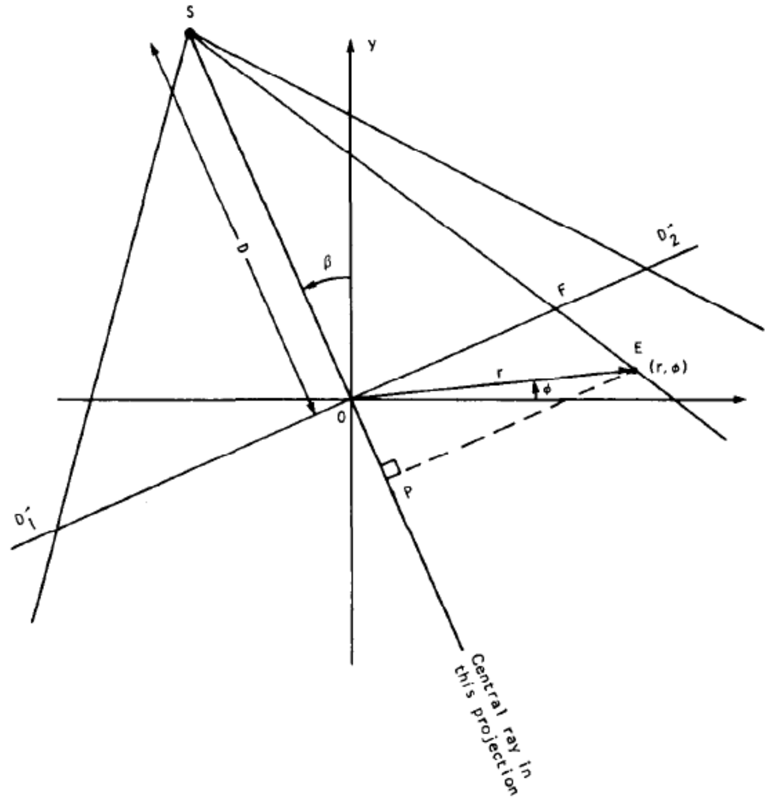


- $R_\beta\ (s)$ is a fan projection and function $f(x,y)$, in polar coordinates $f(r,\phi)$ can be written as:

$$f(r,\phi)=\frac{1}{2}\int_0^{2\pi}\int_{-s_m}^{s_m}R_\beta\ (s)h(r\cos\left(\beta+tan^{-1}\frac{s}{D}-\phi\right)-\frac{Ds}{\sqrt{D^2+s^2}})\frac{D^3}{(D^2+s^2)^{1/2}}\ dsd\beta$$

$$r\ cos(\left(\beta+tan^{-1}\frac{s}{D}-\phi\right)-\frac{Ds}{\sqrt{D^2+s^2}})=r\ cos(\beta-\phi)\frac{D}{\sqrt{D^2+s^2}}-(D+r\ sin\ \ (\beta-\phi))\frac{s}{\sqrt{D^2+s^2}}$$





- R_b (s) is a fan projection and ates $f(r, \phi)$ can be written as:

$$f(r,\phi)=\frac{1}{2}\int_{0}^{2\pi}\int_{-s_m}^{s_m} R_\beta(s)h\left[(s'-s)\frac{UD}{\sqrt{D^2+s^2}}\right]\cdot\frac{D^3}{(D^2+s^2)^{3/2}}\,ds d\beta$$

$$r\,cos\left(\left(\beta+tan^{-1}\frac{s}{D}-\phi\right)-\frac{Ds}{\sqrt{D^2+s^2}}\right)=r\,cos(\beta-\phi)\frac{D}{\sqrt{D^2+s^2}}-(D+r\,sin\,(\beta-\phi))\frac{s}{\sqrt{D^2+s^2}}$$

- The function $h\left[(s' - s)\frac{UD}{\sqrt{D^2+s^2}}\right]$ will be then expressed in terms of $h(t)$. Note that $h(t)$ is the inverse Fourier transform of $|w|$ in the frequency domain:

$$h(t) = \int_{-\infty}^{\infty} |w|\, e^{j2\pi wt}\, dw \longrightarrow h\left[(s' - s)\frac{UD}{\sqrt{D^2+s^2}}\right] = \int_{-\infty}^{\infty} |w|\, e^{j2\pi w(s'-s)(UD/\sqrt{D^2+s^2})}\, dw$$

$$\Downarrow$$

$$w' = w\frac{UD}{\sqrt{D^2+s^2}}$$

- Weighted filtered backprojection algorithm can be written as:

$$\boldsymbol{f(r,\phi) = \int_0^{2\pi} \frac{1}{U^2} \int_{-s_m}^{s_m} R_\beta(s)\, g(s' - s) \frac{D}{\sqrt{D^2+s^2}}\, ds\, d\beta}$$

- where

$$g(s) = \frac{1}{2} h(s)$$

- The image reconstruction is done using the following three steps:
  - Step 1 Each projection $R_\beta(s)$ is sampled with sampling interval a. The known data then are $R_{\beta i}(na)$ where n takes integer values. The first step is to generate for each fan projection $R_{\beta i}(na)$ the corresponding $R'_{\beta i}(na)$:

$$R'_{\beta i}(na) = R_{bi}(na)\,\frac{D}{\sqrt{D^2+n^2a^2}}$$

  - Step 2 Discrete convolution using an FFT program is performed. Each modified projection $R'_{\beta i}(n\alpha)$ with $g(n\alpha)$ is convolved to generate the corresponding filtered projection:

$$Q_{\beta i}(na) = R_{\beta i}(na) \cdot g(na)$$

Response for the discrete impulse →

$$g(na) = \begin{cases} \dfrac{1}{8a^2}, & n=0 \\ 0, & n \text{ even} \\ -\dfrac{1}{2n^2\pi^2a^2}, & n \text{ odd.} \end{cases}$$

  - Step 3 Weighted backprojection of each filtered projection along the fan is performed:

$$f(x,y) = \Delta B \sum_{i=1}^{M} \frac{1}{U^2(x,y,\beta_i)}\, Q_{\beta i}(s')$$

- Algorithm that rapidly re-sorts the fan beam projection data into equivalent parallel beam projection data
- Relationships between the independent variables of the fan beam projections and parallel projections:

$$t = D \sin \gamma \quad and \quad \theta = \beta + \gamma$$

- $R_\beta(\gamma)$ denotes a fan beam projection taken at angle β, and $P_\theta(t)$ a parallel projection taken at angle θ. It can be written:

$$R_\beta(\gamma) = P_{\beta+\gamma}(D \sin \gamma)$$

- β and γ are equal to $m\alpha$ and $n\alpha$, for some integer values of the indices m and n. Therefore it can be written:

$$\boldsymbol{R_{m\alpha}(n\alpha) = P_{(m+n)\alpha}(D\, sin\, n\alpha)}$$

This expresses the fact that the n^{th} ray in the m^{th} radial projection is the n^{th} ray in the $(m+n)^{th}$ parallel projection.

Thank you for your attention

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